Analysis of PR and near-PR cosine-modulated filter banks with odd and even numbers of channels*

XIE Xuemei**, ZHONG Wei and SHI Guangming

(School of Electronic Engineering, Xidian University, Xi'an 710071, China)

Received January 17, 2006; revised April 29, 2006

Abstract It is well known that M-channel filter banks, satisfying either the perfect-reconstruction (PR) or near-PR property, can be obtained by cosine modulation of a linear-phase prototype filter. In this work, we investigated the effects of choosing odd or even number of channels on the performance of both PR and near-PR cosine-modulated filter banks (CMFBs). Given the same m, a performance degradation happens in the PR case with odd numbered M. Contrarily, the choice of even or odd M has little effects on the whole system quality for near-PR one. In addition, we made some comparisons between the constrained optimization and the Parks-McClellan algorithm with cosine roll-off characteristic in designing near-PR CMFBs. Detailed analyses and numerical comparisons show that the PR and near-PR systems have their individual characteristics in terms of M and m selection. Studies here can provide useful and practical guidelines for choosing right filter systems.

Keywords: **cosine modulated filter banks, perfect reconstruction (PR), the Parks-McClellan algorithm.**

Digital filter banks are widely used in a number of signal processing applications, such as subband coders for speech signals, frequency domain speech scramblers, and image coding[1, 2]. Fig. 1 shows a typical structure of M-channel maximally decimated filter banks. Among the proposed filter banks, the cosine modulated filter bank (CMFB) is of particular interest due to its low design cost and low implementation complexity, resulting from the fact that all analysis and synthesis filters in the CMFB are generated by cosine modulation of a linear phase lowpass prototype filter. Two types of CMFBs have been developed so far, pseudo-quadrature mirror filter (QMF) systems[3—6] and perfect reconstruction (PR) systems[7—11]. The former possesses an efficient design procedure, and the latter achieves perfect reconstruction of the input signal, without aliasing, amplitude and phase distortions. Owing to these attractive features, the PR filter banks have received much attention. However, the conventional approaches to the design of PR CMFBs require a complicated nonlinear constrained optimization procedure. Moreover, the filter length is generally constrained to be $2mM$, where M is the number of channels and m is a positive integer. Unlike a PR system, a pseudo-QMF bank, being only approximately free of alias and amplitude distortion, but exactly free of phase distortion, is of near-PR property.

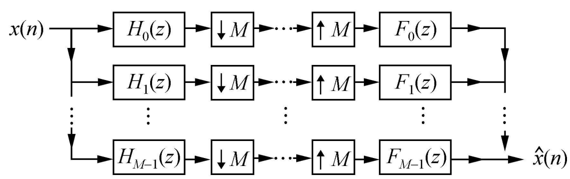

Fig. 1. The structure of M-channel maximally decimated filter banks.

In this paper, we focus on the analysis of the effects of the numbers of channels M on the performance of both PR and near-PR CMFBs to provide a useful guideline for choosing PR or near-PR systems with the appropriate parameters. More specifically, we are interested in discovering how the selection of even or odd numbered M will affect the filter performance. It has been found that in the PR CMFB, if M is odd, two of the polyphase components are forced to be pure delays so that there are $2(m-1)$ zeros contained in the coefficients of prototype filter. It is these zeros that affect the quality of the prototype filter. The larger m is, the more zeros the prototype filter has, and the more difficult the nonlinear optimization becomes. However, in the near-PR CMFB, there are no effects of odd and even numbered M on the performance of the prototype filter.

* Supported by National Natural Science Foundation of China (Grant No. 60372047) and Natural Science Foundation of Shaanxi Province, China (Grant No. 2005F18)

** To whom correspondence should be addressed. E-mail: xmxie@see.xidian.edu.cn

In addition, we made some comparisons between the constrained optimization and the Parks-McClellan algorithm with cosine roll-off characteristic in designing near-PR CMFBs. It is found that by employing the Parks-McClellan algorithm, the CMFB will have much higher stopband attenuation with less design effort.

1 Basic principle of cosine-modulated filter banks

In this section, we briefly introduce the principle of near-PR and PR filter banks constructed by using cosine modulation.

1.1 Near-PR cosine-modulated filter banks

In the CMFBs, the analysis filters $H_k(z)$ and synthesis filters $F_k(z)$ are obtained by using the cosine modulation of a real coefficient linear-phase lowpass prototype filter $H(z)=\sum_{n=0}^{N-1} h(n) z^{-n}$:

$$\begin{aligned} h_k(n) &= 2h(n) \\ &\cdot \cos\left((2k+1)\frac{\pi}{2M}\left(n-\frac{N-1}{2}\right)+(-1)^k\frac{\pi}{4}\right), \\ f_k(n) &= 2h(n) \\ &\cdot \cos\left((2k+1)\frac{\pi}{2M}\left(n-\frac{N-1}{2}\right)-(-1)^k\frac{\pi}{4}\right), \\ &0\leqslant n\leqslant N-1,\quad 0\leqslant k\leqslant M-1, \end{aligned} \tag{1}$$

where $h_k(n)$ and $f_k(n)$ are the impulse responses of $H_k(z)$ and $F_k(z)$, respectively; N is the length of the prototype filter. From (1), we can verify that the analysis and synthesis filters are related by

$$f_k(n) = h_k(N-1-n) \quad \text{and}$$
$$F_k(z) = z^{-(N-1)} H_k(z^{-1}). \tag{2}$$

Therefore, the output of the system shown in Fig. 1 can be expressed in terms of the input as

$$X(z) = X(z)\,T(z) + \underbrace{\sum_{l=1}^{M-1} X(zW_M^l) A_l(z)}_{\text{alias terms}}, \tag{3}$$

where

$$\begin{aligned} T(z) &= \frac{1}{M}\sum_{k=0}^{M-1} H_k(z) F_k(z) \\ &= \frac{1}{M}\sum_{k=0}^{M-1} z^{-(N-1)} H_k(z) H_k(z^{-1}), \\ A_l(z) &= \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW_M^l) F_k(z), \\ W_M &= e^{-j\frac{2\pi}{M}}, \\ &1\leqslant l\leqslant M-1,\quad 0\leqslant k\leqslant M-1. \end{aligned} \tag{4}$$

It can be seen from (4) that $T(z)$, the overall transfer function of the analysis / synthesis system, has linear phase. And the term $X(zW_M^l)$ is a shifted version of $X(z)$ for all l. We refer to the terms, $X(zW_M^l)$ and $A_l(z)$, $1\leqslant l\leqslant M-1$, as alias components and the alias transfer functions, respectively.

Based on the known near-PR filter bank design techniques, it is possible to obtain designs such that the aliasing error and the reconstruction error of the system can be made very small. In this case, the overall transfer function $T(z)$ has approximately unit gain at all frequencies.

1.2 PR cosine-modulated filter banks

Let $\boldsymbol{h}(z)=[H_0(z)\ H_1(z)\ \cdots\ H_{M-1}(z)]^{\mathrm{T}}$ be the analysis filters obtained by cosine modulation of the prototype filter $H(z)$. Using type I polyphase decomposition[2], $\boldsymbol{h}(z)$ can be expressed as

$$\boldsymbol{h}(z) = \boldsymbol{E}(z^M)\boldsymbol{e}_M(z), \tag{5}$$

where $\boldsymbol{E}(z)$ is the polyphase component matrix of the analysis filters and $\boldsymbol{e}_M^{\mathrm{T}}(z)=[1\quad z^{-1}\quad \cdots\quad z^{-(M-1)}]$.

From Ref. [7], we know that, for any PR filter banks, if the analysis and synthesis filters are related by (2), then $\boldsymbol{E}(z)$ is necessarily lossless. Conversely, if the matrix $\boldsymbol{E}(z)$ satisfies $\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)=\boldsymbol{I}_M$, where $\tilde{\boldsymbol{E}}(z)=\boldsymbol{E}^{\mathrm{T}}_*(z^{-1})$, $\forall z$, with superscript T denoting transposition and subscript $*$ the conjugation of coefficients, then the filter bank will possess the PR property. In Ref. [8], the condition on $\boldsymbol{E}(z)$ being lossless is given, where the length of the filter is constrained to be $2mM$ with m being any positive integer. It has been shown that $\boldsymbol{E}(z)$ is lossless if and only if

$$\begin{aligned} \tilde{G}_k(z)G_k(z)+\tilde{G}_{M+k}(z)G_{M+k}(z) &= 1/2M, \\ 0\leqslant k\leqslant M-1, \end{aligned} \tag{6}$$

where $G_k(z)$ are the type I polyphase components of $H(z)$ and $\tilde{G}_k(z)=G_{k*}(z^{-1})$.

With the analysis and synthesis filters obtained by using cosine modulation shown in (1), we just impose the PR constraints (6) on the prototype filter $H(z)$, which is a real coefficient linear phase FIR lowpass filter with cutoff frequency $\pi/2M$. The problem of designing the filter bank is formulated as solving the following constrained optimization problem

$$\min_{h(n)} \Phi = \int_{\omega_s}^{\pi} |H(e^{j\omega})|^2 d\omega \qquad (7)$$

subjected to the PR constraints in (6). The value of the stopband cut-off frequency ω_s depends on the desired transition bandwidth and should be between $\pi/2M$ and π/M. This nonlinear constrained optimization process can be performed using fmincon function available in MATLAB.

2 Analysis of PR cosine-modulated filter banks with odd and even numbers of channels

From the above description, we can see that the CMFB system can achieve perfect reconstruction by constraining the polyphase components of the prototype filter as in (6). However, the length of the filter should be fixed to be $2mM$. Moreover, when M is odd, there are two components being forced to be pure delays, affecting the performance of the prototype filter. The larger m is, the worse the filter quality is. Next, we will give the detailed analysis on this problem.

To begin with, we consider the polyphase components. Due to the linear phase property of the prototype filter $H(z)$, we have

$$G_k(z) = z^{-(m-1)} \tilde{G}_{2M-1-k}(z), \quad 0 \leqslant k \leqslant M-1. \qquad (8)$$

Therefore, almost half of the M constraints given in (6) are redundant. Removing these redundancies, (6) can be expressed as

$$\tilde{G}_k(z) G_k(z) + \tilde{G}_{M+k}(z) G_{M+k}(z) = 1/2M, \quad 0 \leqslant k \leqslant M/2-1, \quad \text{for even } M, \qquad (9a)$$

$$\tilde{G}_k(z) G_k(z) + \tilde{G}_{M+k}(z) G_{M+k}(z) = 1/2M, \quad 0 \leqslant k \leqslant \lfloor M/2 \rfloor - 1,$$
$$2\tilde{G}_{(M-1)/2}(z) G_{(M-1)/2}(z) = 1/2M, \quad k = (M-1)/2, \quad \text{for odd } M. \qquad (9b)$$

The total number of independent constraints is $\lfloor M/2 \rfloor$. From (9), an important observation is obtained: for odd M, the polyphase components $G_{(M-1)/2}(z)$ and $G_{M+(M-1)/2}(z)$ are forced to be pure delays, which would degrade the performance of the prototype filter significantly.

To prove the claim above, we will give two numerical examples with even and odd numbers of channels. Before doing that, we need to define the following two quantities to measure the PR property of the filter banks.

1) The peak-to-peak reconstruction error E_{pp}: The maximum peak to peak ripple of $M|T(e^{j\omega})|$, denoted by E_{pp}, is usually taken to be a measure of worst possible amplitude distortion.

2) The aliasing error E_a: $E_a = \max_{\omega} E(\omega)$, where $E(\omega) = \left[\sum_{l=1}^{M-1} |A_l(e^{j\omega})|^2\right]^{1/2}$. It is a measure of the worst possible peak aliasing distortion.

As a general rule, if the values of E_{pp} and E_a are in the order of 10^{-3} or below, we consider the corresponding filter bank to be a near-PR one; on the other hand, if both values are in the order of 10^{-12} or below, we classify the filter bank as a PR one.

Example with even M: In this example, we choose $M=4$, $m=13$, and $N=2mM=104$. And we set the cut-off frequency $\omega_c=\pi/2M$. Following Eq. (7) under the constraints (6), we obtain the prototype filter $h(n)$ by using fmincon function in MATLAB and a set of the analysis and synthesis filters by applying Eq. (1). Fig. 2 shows the frequency responses of the analysis filters of this 4-channel filter bank. E_{pp} and E_a are also shown in this figure. They are approximately in the order of 10^{-15} and 10^{-16}, respectively, being considerably below the order of 10^{-12} to satisfy the PR requirement. The stopband attenuation A_s is 82.1 dB.

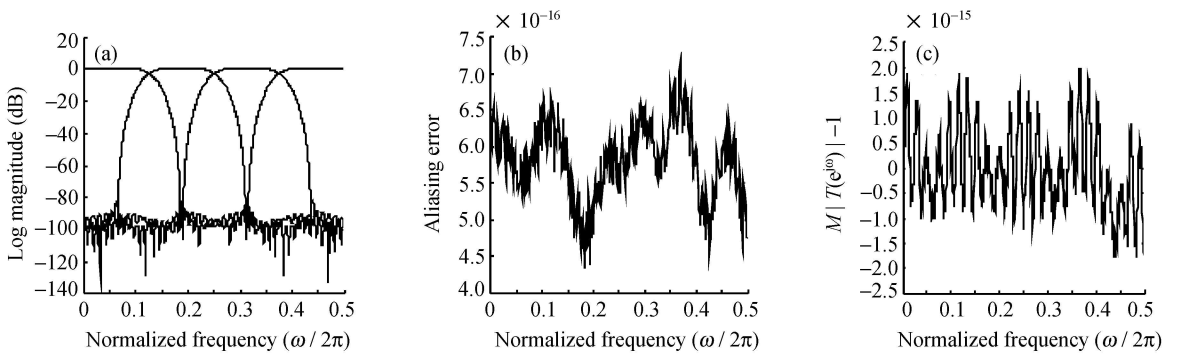

Fig. 2. A 4-channel PR CMFB. (a) Magnitude responses of the analysis filters; (b) aliasing error; (c) amplitude distortion.

Example with odd M: For the purposes of comparison, we give the example with odd number of channels. Here, we consider the case that $M=5$, $m=13$, $N=2mM=130$, and $\omega_c=\pi/2M$. Following the same procedure as the above, we got the analysis filters of the 5-channel filter bank. Fig. 3 shows the frequency responses of the analysis filters as well as E_{pp} and E_a, which are approximately in the order of 10^{-14} and 10^{-15}, respectively, again being significantly below the order of 10^{-12}. The stopband attenuation A_s is only 41.41 dB, which is about 40 dB worse than that of the 4-channel case while having almost the same E_{pp} or E_a. These values are listed in Table 1.

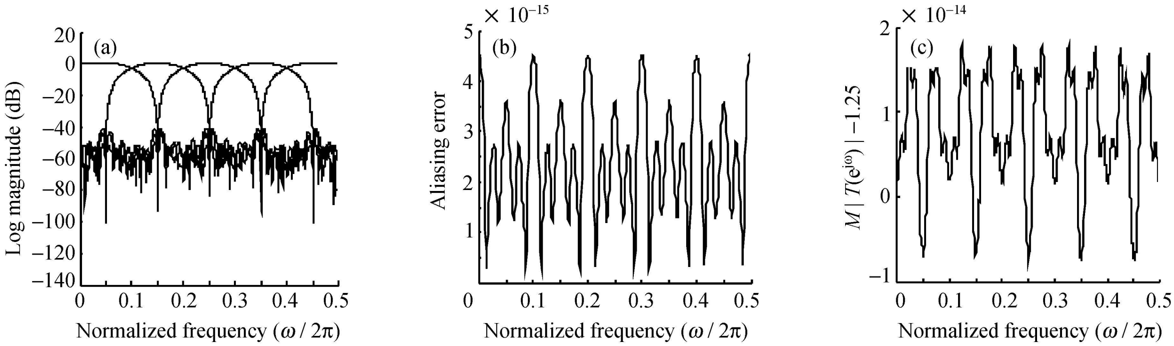

Fig. 3. A 5-channel PR CMFB. (a) Magnitude responses of the analysis filters; (b) aliasing error; (c) amplitude distortion.

Table 1. Comparison between the odd/even M in PR case ($m=13$, $\omega_c=\pi/2M$)

Number of channels	A_s(dB)	Reconstruction error (E_{pp})	Aliasing error (E_a)
$M=4$	82.10	3.997×10^{-15}	7.202×10^{-16}
$M=5$	41.41	2.554×10^{-14}	4.545×10^{-15}

This performance degradation phenomenon can be easily explained based on our analysis. As mentioned above, in the case that M is odd, the polyphase components $G_{(M-1)/2}(z)$ and $G_{M+(M-1)/2}(z)$ have to be pure delay elements to maintain the PR property. There would be $2(m-1)$ zero coefficients of the prototype filter. This reduces the degree of the freedom in the optimization, and hence affects the quality of the prototype filter. It is obvious that the larger m is, the more zero coefficients the prototype filter has, and the more difficult the nonlinear optimization becomes. Therefore, when M is odd, a small m should be chosen. It can be expected that good performance of the filter bank with odd M can be obtained when M is large and m is small.

3 Analysis of near-PR filter banks and their design approaches

Compared with PR filter banks, the difference of near-PR filter banks with odd or even M is not so obvious. In this section, we will address this phenomenon by giving examples in detail. In addition, two different methods with the constrained optimization and the Parks-McClellan algorithm for designing the near-PR CMFBs will be compared.

3.1 Near-PR filter banks with odd and even numbers of channels

There are many methods available for the design of near-PR filter banks. The design is usually performed by using the nonlinear optimization with the relaxed PR constraints. Since optimization procedures are in general computationally intensive and the converging rate towards optimum solution is rather slow, we employ the Parks-McClellan algorithm with cosine roll-off characteristic in constructing the prototype filter. An ideal magnitude response can be achieved without nonlinear optimization[5].

Based on the characteristics of the CMFB, there is no phase distortion and no significant aliasing error in the filter bank system. The only remaining issue is to reduce amplitude distortion. It is known that there would be no amplitude distortion arising if $|T(e^{j\omega})|$ is kept to be flat. In designing near-PR filter banks, the prototype filter $H(z)$ should be designed in such a way that $|T(e^{j\omega})|$ is acceptably flat. If we guarantee that the stopband attenuation of $H_k(z)$ is sufficiently high, then $|T(e^{j\omega})|$ will be nearly as flat as the magnitude response of each filter $H_k(z)$ in the region with ω allocated at the passbands of all $H_k(z)$. Therefore, to meet the requirements described above, it is sufficient to impose the following constraint on the prototype filter $H(z)$:

$$|H(e^{j\omega})|^2 + |H(e^{j(\omega-\pi/M)})|^2 = 1,\quad 0 < \omega < \pi/M. \tag{10}$$

It is found that if the response of $H(z)$ follows a cosine function in its transition band, then the filter bank is approximately power complementary and hence has the near-PR property. For simplicity, we use the Parks-McClellan algorithm with cosine roll-off characteristic in designing the prototype filter, and an ideal magnitude response can be achieved. This is performed by using remez (or firpm) function in MATLAB.

In section 2, we have seen the effect with even or odd number of channels on the performance of the PR CMFBs. In this section, we will show the effect on that of the near-PR CMFBs by using the following two examples.

Example with even M: We will consider the case where $M=4$, $m=13$ and $N=104$, the same parameters as those used in the example of section 2. Following the above design procedures, the near-PR filter bank was obtained. The frequency responses of the analysis filters are shown in Fig. 4. E_{pp} and E_a are also shown in the figure. E_{pp} and E_a are respectively in the order of 10^{-3} and 10^{-9}, satisfying the near-PR requirement. A_s is 160.12 dB. All the values are listed in Table 2.

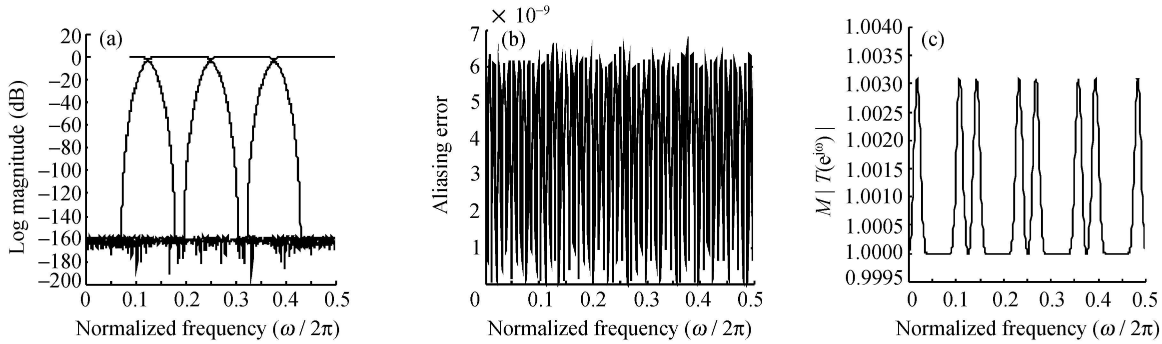

Fig. 4. A 4-channel near-PR CMFB. (a) Magnitude responses of the analysis filters; (b) aliasing error; (c) amplitude distortion.

Table 2. Comparison between the odd/even M in near-PR case ($m=13$, $\omega_c=\pi/2M$)

Number of channels	A_s(dB)	Reconstruction error (E_{pp})	Aliasing error (E_a)
$M=4$	160.12	3.094×10^{-3}	6.534×10^{-9}
$M=5$	157.79	2.390×10^{-3}	1.248×10^{-9}

Example with odd M: We also used the same parameters as those in the second example of section 2, i.e. $M=5$, $m=13$, and $N=130$. Fig. 5 shows the frequency responses of the derived analysis filters. The corresponding performance indexes are listed in Table 2. In this example, A_s is 157.79 dB.

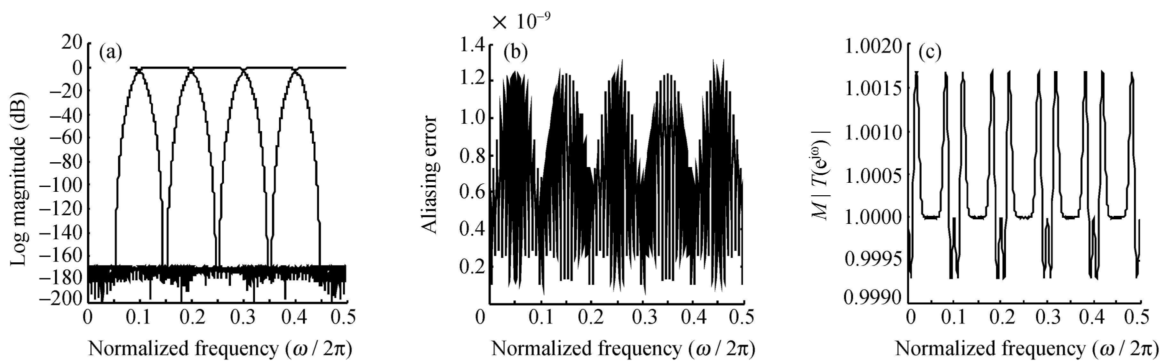

Fig. 5. A 5-channel near-PR CMFB. (a) Magnitude responses of the analysis filters; (b) aliasing error; (c) amplitude distortion.

It can be seen from Table 2 that in the case of near-PR filter banks, there is no noticeable difference in terms of the stopband attenuation of the filter banks between even and odd M while maintaining similar E_{pp} and E_a. In other words, the selection of even or odd number of channels has little effects on the performance of the filter banks. Moreover, compared with their PR counterparts, the stopband attenuation of the prototype filter for near-PR CMFBs is much higher at the cost of sacrificing the PR property. In addition, there is no constraint on the length of the prototype filter in designing near-PR CMFBs by using the proposed method. This is because there are no PR constraints on the polyphase components of the filter.

3.2 Near-PR filter banks by using different design approaches

Compared with the conventional method for designing near-PR filter banks, where the nonlinear optimization is employed with relaxing the PR constraints, the Parks-McClellan algorithm with cosine roll-off characteristic has some distinctive advantages. For easy comparison, we will present an example by using the conventional method.

Example with the constrained optimization: We consider the same parameters used in the example given in Section 3.1, where $M=4$, $m=13$, and $N=2mM=104$. We can get a near-PR CMFB by relaxing the PR constraints in (6) while using the nonlinear optimization as in (7). Fig. 6 shows the frequency responses of the analysis filters of this 4-channel filter bank. E_{pp} and E_a are also shown in the figure. E_{pp} and E_a are in the order of 10^{-4} and 10^{-7}, respectively, satisfying the near-PR requirement. A_s is about 107.55 dB. A_s, E_{pp} and E_a are listed in Table 3.

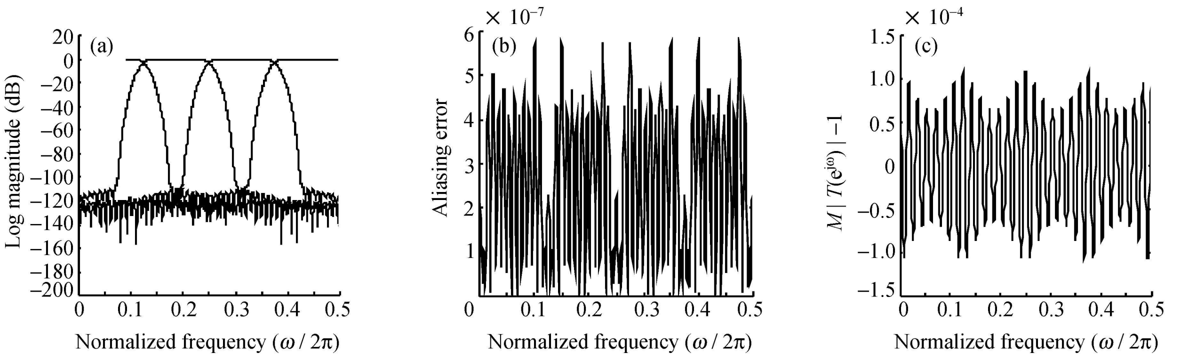

Fig. 6. A 4-channel near-PR CMFB by using the constrained optimization. (a) Magnitude responses of the analysis filters; (b) aliasing error; (c) amplitude distortion.

Table 3. Comparison between the constrained optimization and the Parks-McClellan algorithm with the same parameters

Design methods	A_s(dB)	Reconstruction error (E_{pp})	Aliasing error (E_a)
Nonlinear optimization (relaxed PR constraints)	107.55	2.151×10^{-4}	5.728×10^{-7}
Parks-McClellan algorithm	160.12	3.094×10^{-3}	6.534×10^{-9}

Comparing this example with the one in Section 3.1, it can be found that under the similar E_{pp} and E_a, the stopband attenuation by using the Parks-McClellan algorithm is much higher (160.12 dB versus 107.55 dB). For the constrained optimization, we minimize the objective function in the least squares sense as shown in (7), and therefore the prototype filter is optimal in the least squares sense. Even if the prototype filter is equiripple and optimal in the minimax sense after the nonlinear optimization, its stopband attenuation is not comparable with that obtained by using the Parks-McClellan algorithm. This is because the Parks- McClellan algorithm is an optimal design in the minimax sense with respect to the specified ideal frequency response of the prototype filter. Most importantly, however, the given examples have shown that the Parks-McClellan algorithm is very simple and efficient.

4 Conclusion

In this paper, an issue of choosing even and odd numbers of channels in designing PR CMFBs is identified. We pay particular attention to the analysis of the effects of odd and even numbers of channels on the performance of filter banks having PR or near-PR property. Furthermore, in the near-PR case, the comparison between the constrained optimization and the Parks-McClellan algorithm with cosine roll-off characteristic is made. By detailed analysis and numerical comparisons, we concluded that the PR and the near-PR systems have their individual characteristics in terms of M and m selection. The conclusions and recommendations made in this paper can provide some useful and practical guidelines for choosing right filter systems.

References

1 Vaidyanathan P. P. Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial. Proceedings of the IEEE, 1990, 78(1): 56—93.

2 Vaidyanathan P. P. Multirate Systems and Filter Banks. Englewood Cliffs, NJ: Prentice Hall, 1993.

3 Nguyen T. Q. Near-prefect-reconstruction pseudo-QMF banks. IEEE Trans. on Signal Processing, 1994, 42(1): 65—75.

4 Nussbaumer H. Pseudo QMF filter bank. IBM Technical Disclosure Bulletin, 1981, 24(6): 3081—3087.

5 Xie X. M., Shi G. M. and Chen X. Y. Optimal prototype filters for near-perfect-reconstruction cosine-modulated filter banks. CIS 2005, Part II, LNAI 3802, 2005, 851—856.

6 Furtado M. B. J., Diniz P. S. R. and Netto S. L. Numerically efficient optimal design of cosine-modulated filter banks with peak-constrained least-squares behavior. IEEE Trans. on Circuits and Systems I, 2005, 52(3): 597—608.

7 Vaidyanathan P. P. Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having the perfect reconstruction property. IEEE Trans. on Acoust., Speech, Signal Processing, 1987, 35(4): 476—492.

8 Koilpillai R. D. and Vaidyanathan P. P. Cosine-modulated FIR filter banks satisfying perfect reconstruction. IEEE Trans. on Signal Processing, 1992, 40(4): 770—783.

9 Nguyen T. Q. and Koilpillai R. D. The theory and design of arbitrary-length cosine-modulated filter banks and wavelets, satisfying perfect reconstruction. IEEE Trans. on Signal Processing, 1996, 44(3): 473—483.

10 Lu W. S., Saramäki T. and Bregovic R. Design of practically perfect-reconstruction cosine-modulated filter banks: A second-order cone programming approach. IEEE Trans. on Circuits and Systems I, 2004, 51(3): 552—563.

11 Xie X. M., Chan S. C. and Yuk T. I. On the design of a class of PR nonuniform cosine modulated filter banks with flexible rational sampling. IEEE Trans. on Circuits and Systems I, 2005, 52(9): 1965—1981.